Signal Analysis & Communication ECE 355

Ch. 2.1: DT LTI Systems

Lecture 7

21-09-2023

Ch. 2 LINEAR TIME INVARIANT (LTI) SYSTEMS

- Linearity & time invariance play a fundamental role in sig. & sys. analysis as:
  1. Many physical processes possess these properties & thus can be modelled as LTI systems
  2. Many useful engg. algorithms are described by LTI systems.
  3. We have good theoretical tools for analyzing LTI systems.
  4. We have good procedures available for designing LTI systems.

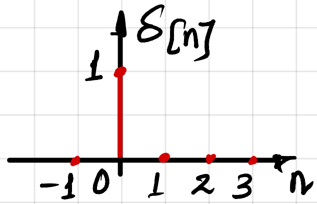
* You will use LTI systems in many other courses: control, communications, signal processing, robotics ...

RECALL
- Linearity means "the superposition principle holds."
- Time-invariance means "the sys. will be the same tomorrow as it is today."

Ch. 2·1 DT LTI SYSTEMS

- Recall the DT unit impulse & the sifting formula

$$\delta[n] = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n\neq 0 \end{cases}$$



$$x[k] = \sum_{n=-\infty}^{\infty} x[n]\,\delta[n-k]$$

OR

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \quad — ①$$

$\Rightarrow$ Any DT sig. can be written as eqn ①

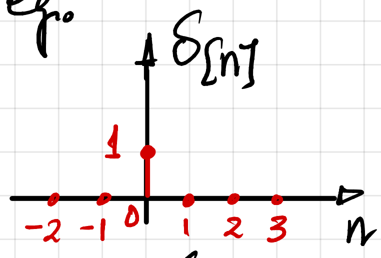
- Now let's define "impulse response", $h[n]$, is the response to a DT LTI system to a unit impulse, $\delta[n]$, I/p applied at $n=0$.

$$\delta[n] \xrightarrow{S} h[n]$$

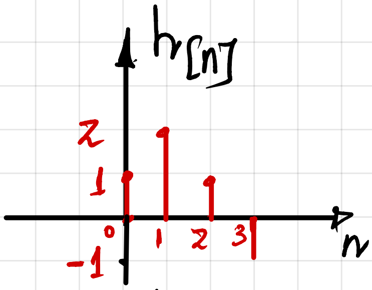
- Then time-invariance implies that

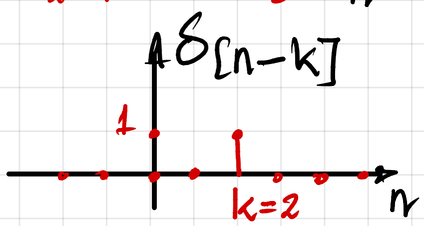
$$\delta_{[n-k]} \xrightarrow{S} h_{[n-k]} \qquad \forall k \in \mathbb{Z}$$

For eg.

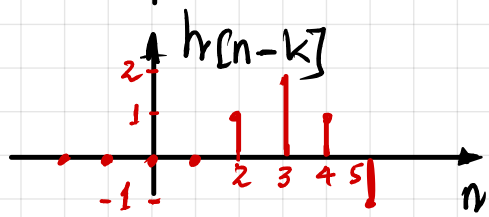


$\xrightarrow{S}$



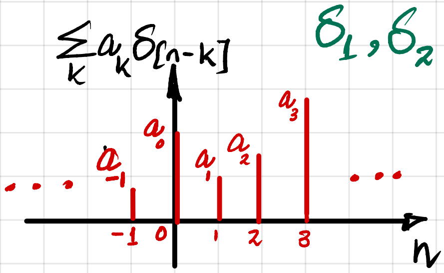


$\xrightarrow{S}$

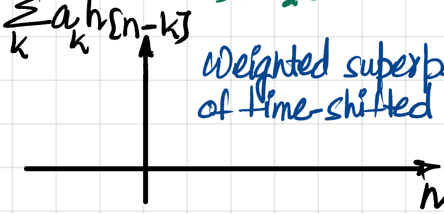


- And the linearity implies that

$$\sum_k a_k \underbrace{\delta_{[n-k]}}_{\delta_1,\,\delta_2,\,\dots} \xrightarrow{S} \sum_k a_k \underbrace{h_{[n-k]}}_{h_1,\,h_2,\,\dots}, \quad \forall \{a_k\} \quad \text{—} (2)$$



$\xrightarrow{S}$



Weighted superposition of time-shifted $h_{[n]}$

- Applying eqn (2) to eqn (1), we will obtain the following theorem.

THEOREM: For a DT LTI system with impulse response $h_{[n]}$, the output $y_{[n]}$ for an input $x_{[n]}$ is given as:

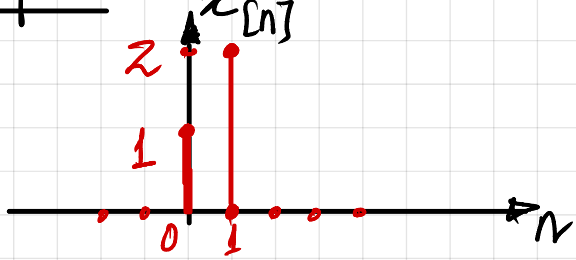
$$x_{[n]} = \sum_{k=-\infty}^{\infty} x_{[k]}\,\delta_{[n-k]} \xrightarrow{S} \boxed{y_{[n]} = \sum_{k=-\infty}^{\infty} x_{[k]}\,h_{[n-k]}} \quad (3)$$

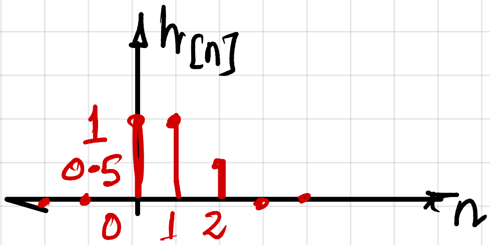
Proof As above.

$$\boxed{y_{[n]} \triangleq x_{[n]} * h_{[n]}}$$

DT Convolution

Example 1





$y_{[n]} = ?$

- Applying eqn (3) to find $y_{[n]}$
- Two interpretations:

(a) Weighted superposition of time-shifted signals $h_{[n-k]}$

$k<0 \qquad x_{[k]} = x_{[n]} = 0$

$k=0 \qquad x_{[0]} = 1$

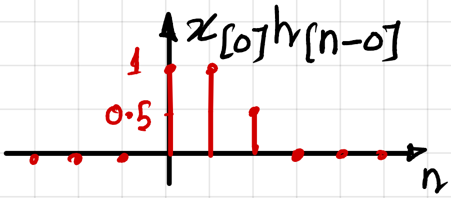
$k=1 \qquad x_{[1]} = 2$

$k>1 \qquad x_{[k]} = x_{[n]} = 0$

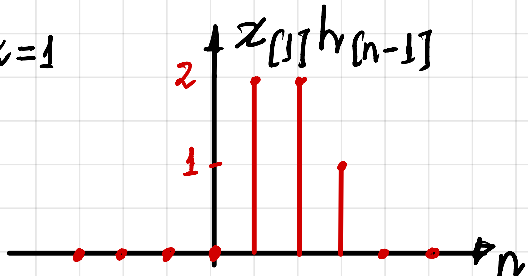
$\therefore$ eqn (3) $\Rightarrow$ $y_{[n]} = x_{[0]} h_{[n-0]} + x_{[1]} h_{[n-1]}$

$= h_{[n]} + 2 h_{[n-1]}$

$k=0$



$k=1$



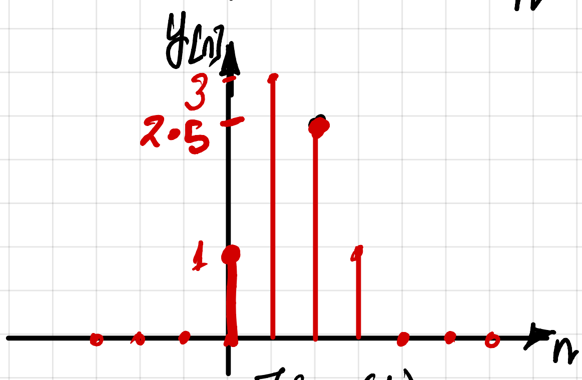
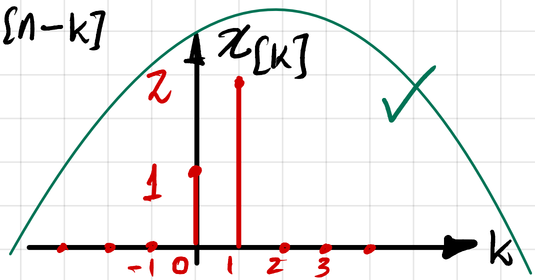


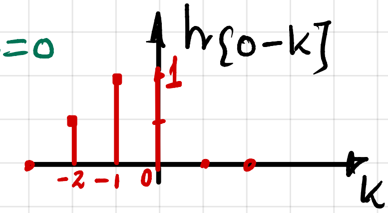
Fig. (1)

(b) Sum in dimension k of $x_{[k]} h_{[n-k]}$



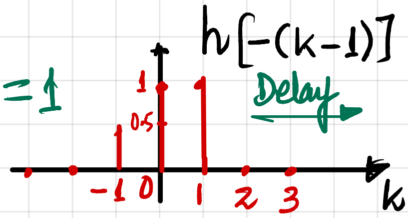
$$y[0] = \sum_{k=-\infty}^{\infty} x[k]\,h[0-k] = 1$$

$n=0$



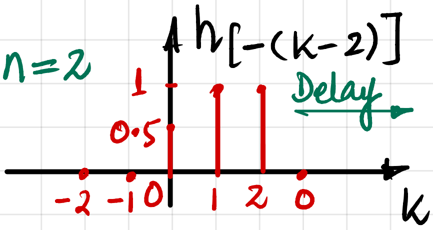
$$y[1] = \sum_{k=-\infty}^{\infty} x[k]\,h[1-k] = 1+2 = 3$$

$n=1$



$$y[2] = \sum_{k=-\infty}^{\infty} x[k]\,h[2-k] = 0.5+2 = 2.5$$

$n=2$



Similarly

$$y[3] = \sum_{k=-\infty}^{\infty} x[k]\,h[3-k] = 1$$

For $n<0$ or $n>3$, $x[k]\,h[n-k] = 0,\ \forall k$, so $y[n] = 0$

We get the same Fig. (1) by plotting $y[n]$ for $n = 0, 1, 2, 3$.

Summary: To perform Convolution of $x[n]$ & $h[n]$

I. Have/plot $x[k] = x[n]$

$h[k] = h[n]$

Depending on the problem can do the other way round Conv. is Commutative (NEXT)

II. Flip $h[k]$: $h[-k]$

III. Slide $h[-k]$ to the right to get $h[n-k]$ → For all the possible values of 'n' where product in IV is not zero.

IV. Multiply $h[n-k]$ by $x[k]$ and sum.

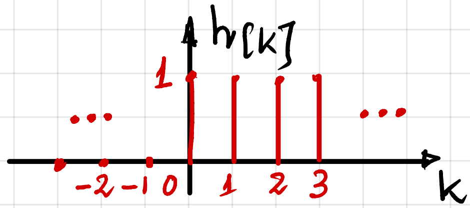
Example 2

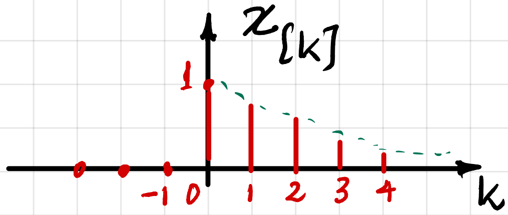
$$h[n] = u[n]$$

$$x[n] = \alpha^n u[n], \quad 0 < \alpha < 1$$

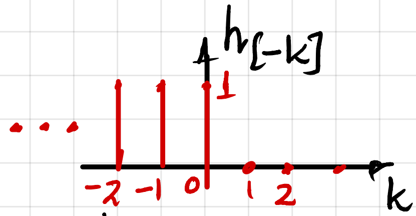
Find $y[n] = x[n] * h[n]$

Step I.

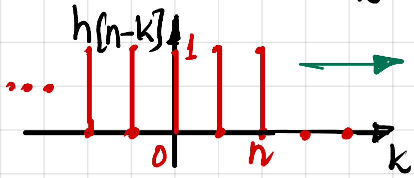




Step II. Flip $h_{[k]}$: $h_{[-k]}$



Step III. Slide to the right: $h_{[n-k]}$



Step IV. Multiply by $x_{[k]}$ & sum

For $n<0$: $y_{[n]}=\sum_{k=-\infty}^{\infty} x_{[k]}\, h_{[n-k]}=0$ | No overlap b/w $x_{[k]}$ and $h_{[n-k]}$, their product is zero!

For $n \geq 0$:

$$y_{[n]}=\underbrace{\sum_{k=0}^{\infty}}_{x_{[k]}=0 \text{ for } k<0} \alpha^k \times 1 = \frac{1-\alpha^{n+1}}{1-\alpha}$$

Therefore,

$$y_{[n]}=\frac{1-\alpha^{n+1}}{1-\alpha}\, u_{[n]}$$

indicates that $y_{[n]}$ exists for $n \geq 0$.